The effect of eddy viscosity on the velocity profile of steady flow in a uniform rough channel

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(Received 28 August 1958)

The equation of motion is derived for the steady flow under gravity of a liquid in a uniform channel, including the effects of the friction of the bed and of the eddy viscosity for lateral mixing. The equation is solved to find the distribution of mean velocity across the channel when the cross-section is rectangular, triangular or trapezoidal. Numerical values are also given to indicate the extent to which eddy viscosity may affect the lateral distribution of mean velocity.

1. Introduction

We consider the steady flow of a river or channel with a straight course and having a bed which is uniform in its direction of flow, and which has the same downward slope at all its points. We take the x-axis in the downward direction o flow, and the y-axis horizontal and perpendicular to the direction of flow. Let u denote the mean velocity of the fluid particles lying on the straight line through any point of the bed parallel to the z-axis. The bed will exert a force on the moving fluid, and we shall assume that the force on a cylindrical column of fluid standing on area $dx\,dy$ with generators parallel to the z-axis may be represented by $-f\,|u|\,u\,dx\,dy$, where f is regarded as the coefficient of friction of the bed. This assumption is usually known in hydraulics as the Chézy formula, and references about it may be found in Stoker (1957, p. 466). The broad argument behind the Chézy formula is that the momentum of the fluid is destroyed when it is brought into contact with the bed by the eddies. The momentum of the column is proportional to u, and we assume that the speed of the eddies is proportional to $|u|$. The rate of destruction of momentum will be proportional to the momentum of the column and to the rate at which it is brought into contact with the bed, and hence can be represented by the term $f\,|u|\,u\,dx\,dy$, where f is a constant of proportionality.

The mixing process which brings fluid in contact with the bed will also cause lateral mixing. Since the mean velocity of flow will be a function of y, the mixing will give rise to viscous effects, and we assume that the effect is represented by ν, the kinematic coefficient of eddy viscosity. In the two following sections we shall regard ν as a constant, but in later sections we shall make a different assumption which allows for the variation of ν with the variation in depth and mean velocity. The inclination α of the bed to the horizon is assumed to be small, so that $\cos\alpha = 1$ to a sufficient approximation, and we write $\sin\alpha = s$. We shall also make the

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assumption that the pressure at any point is approximately the same as the hydrostatic pressure, which is proportional to the depth below the effective surface.

Now consider the motion of a column of fluid standing on area $dx\,dy$. We assume that all points of the column are moving in the x-direction with velocity u, which is the mean velocity of the particles of the column. The forces acting on the column in the x-direction are: (i) $-fu^2\,dx\,dy$ being the frictional force of the bed, (ii) $gsh\,dx\,dy$ due to gravity, and (iii) $\dfrac{d}{dy}\left(h\nu\dfrac{du}{dy}dx\right)dy$ due to eddy-viscosity. If the motion is uniform, i.e. if there is no variation with x, the acceleration of the column will be zero, and the component of the force due to pressure will also be zero; therefore the equation of motion will be

$$0 = -fu^2 + gsh + \frac{d}{dy}\left(h\nu\frac{du}{dy}\right). \tag{1}$$

To obtain the Chézy formula, the last term on the right-hand side of (1) is neglected, which gives

$$u^2 = \frac{ghs}{f}. \tag{2}$$

The limitations of formula (2) are that (i) it neglects the boundary conditions at the walls of the channel where the mean velocity is necessarily zero, and that (ii) if dh/dy is discontinuous at any point (e.g. in a triangular channel) du/dy will also be discontinuous, implying an infinite viscous force at that point. These defects arise from the neglect of the last term in (1) in regions where it is important. We shall investigate the results when this term is included.

2. The velocity profile in an infinitely wide rectangular channel with constant eddy viscosity

We consider an infinitely wide rectangular channel, i.e. a channel in which $h = H$, a constant. If eddy viscosity is neglected, the velocity is given by

$$U^2 = \frac{gHs}{f}. \tag{3}$$

We shall seek a solution of equation (1) with constant ν satisfying the boundary conditions $u = 0$ at $y = 0$ and $u = U$, $du/dy = 0$ at $y = \infty$. Substitution of (3) in (1) gives

$$\frac{d^2u}{dy^2} + \frac{f}{H\nu}(U^2 - u^2) = 0, \tag{4}$$

which has the integral

$$\left(\frac{du}{dy}\right)^2 = \frac{2f}{3H\nu}(U-u)(2U^2 - Uu - u^2), \tag{5}$$

the constant of integration being determined from the condition that $du/dy = 0$ when $u = U$.

By the substitution $v = 2U + u$ in (5) it is easy to perform a second integration, and the result can be expressed in the form

$$\frac{u}{3U} = \left\{\frac{\sqrt{3}(e^{ky}-1)+\sqrt{2}(e^{ky}+1)}{\sqrt{3}(e^{ky}+1)+\sqrt{2}(e^{ky}-1)}\right\}^2 - \tfrac{2}{3}, \tag{6}$$

where $k = \sqrt{(2fU/H\nu)}$ and the boundary condition at $y = 0$ has been used to eliminate the constant of integration.

Table 1 gives some values of u/U against ky according to equation (6). From the illustrative values of f and ν that will be considered in § 8, it is seen that k is of the order of the reciprocal of H, so that the effect of eddy-viscosity in reducing the mean velocity is practically confined to a distance equal to a few times the depth.

ky	0	0·2	0·4	0·8	2	4
u/U	0	0·153	0·228	0·502	0·716	0·998

TABLE 1

3. Velocity profile of a rectangular river of finite width with constant eddy viscosity

Let 2η be the breadth of the river. In this case we need a solution of (4) with the initial values $u = 0$ at $y = 0$ and $du/dy = 0$ at $y = \eta$.

Let $u = u_*$ at $y = \eta$; then the first integration of (4) gives

$$\left(\frac{du}{dy}\right)^2 = \frac{2f}{3H\nu}(u_* - u)(3U^2 - u_*^2 - uu_* - u^2).$$

Putting $z = u_* - u$, the above equation reduces to

$$\left(\frac{dz}{dy}\right)^2 = -\frac{2f}{3H\nu}(z-\alpha)(z-\beta)(z-\gamma), \tag{7}$$

where

$$\left.\begin{aligned} \alpha &= \tfrac{3}{2}u_* + (3U^2 - \tfrac{3}{4}u_*^2)^{\frac{1}{2}} \quad (\beta = 0), \\ \gamma &= \tfrac{3}{2}u_* - (3U^2 - \tfrac{3}{4}u_*^2)^{\frac{1}{2}}. \end{aligned}\right\} \tag{8}$$

Equation (7) can be integrated in terms of elliptic functions (e.g. see Milne-Thomson 1950, p. 29); and using the boundary condition that $z = 0$ at $y = \eta$, we get

$$\int_\eta^y -\left(\frac{2f}{3H\nu}\right)^{\frac{1}{2}} dy = \int_0^z \{-(z-\alpha)(z-\beta)(z-\gamma)\}^{-\frac{1}{2}} dz$$

$$= \lambda\,\mathrm{dn}^{-1}\left[\left\{\frac{(3U^2 - \frac{3}{4}u_*^2)^{\frac{1}{2}} - \frac{3}{2}u_*}{z + (3U^2 - \frac{3}{4}u_*^2)^{\frac{1}{2}} - \frac{3}{2}u_*}\right\} \Big/ m_1\right], \tag{9}$$

where

$$\lambda = 2^{\frac{1}{2}}(3U^2 - \tfrac{3}{4}u_*^2)^{-\frac{1}{4}} \tag{10}$$

and

$$m_1 = \frac{(3U^2 - \frac{3}{4}u_*^2)^{\frac{1}{2}} + \frac{3}{2}u_*}{2(3U^2 - \frac{3}{4}u_*^2)^{\frac{1}{2}}}. \tag{11}$$

On simplification (9) yields

$$\left\{\frac{(3U^2 - \frac{3}{4}u_*^2)^{\frac{1}{2}} - \frac{3}{2}u_*}{(3U^2 - \frac{3}{4}u_*^2)^{\frac{1}{2}} - u - \frac{1}{2}u_*}\right\}^{\frac{1}{2}} = \mathrm{dn}\left\{\left(\frac{f}{3H\nu}\right)^{\frac{1}{2}}(3U^2 - \tfrac{3}{4}u_*^2)^{\frac{1}{4}}(\eta - y)\Big/ m_1\right\}, \tag{12}$$

from which u_* can be found by use of the condition that $u = 0$ at $y = 0$.

4. Velocity profile in an infinitely wide rectangular channel with eddy viscosity varying linearly with depth and mean velocity

We have so far assumed the kinematic coefficient of eddy viscosity to be constant, but it is perhaps more realistic to assume that eddy viscosity depends on the depth and the mean velocity. The depth of the channel will act as a limiting factor on the size of the eddies, and also we expect more rapidly moving eddies where the mean velocity is greater. The simplest assumption to allow for these facts is to assume that the eddy viscosity varies linearly with depth and mean velocity, and we write

$$\nu = cHu, \tag{13}$$

where H is the uniform depth and c is a dimensionless constant.

Substituting (3) and (13) in (1), we get

$$\frac{d^2u^2}{dy^2}+\frac{2f}{cH^2}(U^2-u^2) = 0, \tag{14}$$

which is a linear equation with constant coefficients in u^2 and the solution of this equation which satisfies the boundary conditions $u = 0$ at $y = 0$ and $u = U$ at $y = \infty$ is

$$u^2 = U^2\{1-e^{-(2f/cH^2)^{\frac{1}{2}}y}\}. \tag{15}$$

5. Velocity profile of a rectangular channel of finite width with eddy viscosity varying linearly with depth and mean velocity

We take the breadth to be 2η and seek a solution of (14) with the boundary conditions that $u = 0$ at $y = 0$ and $y = 2\eta$. The solution is

$$u^2 = U^2\left[1-\frac{\cosh\{(2f/cH^2)^{\frac{1}{2}}(\eta-y)\}}{\cosh\{(2f/cH^2)^{\frac{1}{2}}\eta\}}\right]. \tag{16}$$

The maximum velocity u_* at $y = \eta$ is given by

$$u_*^2 = U^2[1-\operatorname{sech}\{(2f/cH^2)^{\frac{1}{2}}\eta\}]. \tag{17}$$

6. Velocity profile of a triangular channel with eddy viscosity varying linearly with depth and mean velocity

We have so far assumed the depth to be constant. We now consider a channel whose cross-section has the form of an isosceles triangle of base angle $\tan^{-1}\kappa$. The depth at a distance y from the bank is given by $h = \kappa y$ if $y \leqslant \eta$, and $h = \kappa(2\eta-y)$ if $\eta \leqslant y \leqslant 2\eta$, where 2η is the breadth of the channel. We need consider only the flow in the region $0 \leqslant y \leqslant \eta$, the flow in the other half of the channel being given by symmetry.

Using $\nu = chu$, it is easily seen that the equation of motion (1) reduces in this case to the form

$$y^2\frac{d^2u^2}{dy^2}+2y\frac{du^2}{dy}-\frac{2f}{\kappa^2c}u^2 = -\frac{2gs}{\kappa c}y. \tag{18}$$

The solution of this equation satisfying the boundary conditions $u = 0$ at $y = 0$ and $du/dy = 0$ at $y = \eta$ is

$$u^2 = \frac{gs\kappa}{f-\kappa^2 c}\left[y - \frac{\eta}{(\frac{1}{4}+2f/\kappa^2 c)^{\frac{1}{2}}-\frac{1}{2}}\left(\frac{y}{\eta}\right)^{(\frac{1}{4}+2f/\kappa^2 c)^{\frac{1}{2}}-\frac{1}{2}}\right] \tag{19}$$

if $f \neq \kappa^2 c$, and

$$u^2 = \frac{2gs}{3\kappa c}[y(1+\log\eta) - y\log y] \tag{20}$$

if $f = \kappa^2 c$.

7. Velocity profile in a trapezoidal channel with eddy viscosity varying linearly with depth and mean velocity

We have found solutions for a rectangular channel in § 5, and for a triangular channel in § 6. It is possible to combine the equations obtained in these two sections to find solutions for a trapezoidal channel. Assume that in the channel of width 2η, the depth is given by $h = \kappa y$ for $0 \leqslant y \leqslant \eta_1$, $h = H = \kappa\eta_1$ for $\eta_1 \leqslant y \leqslant 2\eta - \eta_1$, and $h = \kappa(2\eta - y)$ for $(2\eta - \eta_1) \leqslant y \leqslant 2\eta$. The equation of motion will be (18) for the regions $0 \leqslant y \leqslant \eta_1$ and $(2\eta - \eta_1) \leqslant y \leqslant 2\eta$, and will be (14) for the region $\eta_1 \leqslant y \leqslant 2\eta - \eta_1$.

We take the general solutions of (18) and (14), and apply the boundary conditions: (i) $u = 0$ at $y = 0$; (ii) u is continuous at $y = \eta_1$; (iii) $du/dy = 0$ at $y = \eta$; (iv) du/dy is continuous at $y = \eta_1$.

Hence, the solution for the region $0 \leqslant y \leqslant \eta$ (the solution for the other half is given by symmetry) is found to be

$$u^2 = \frac{gs\kappa y}{f-\kappa^2 c} + Ay^{(\frac{1}{4}+2f/\kappa^2 c)^{\frac{1}{2}}-\frac{1}{2}}, \tag{21}$$

and the solution for $\eta_1 \leqslant y \leqslant \eta$ is

$$u^2 = \frac{gsH}{f} + 2A'\, e^{(2f/cH^2)^{\frac{1}{2}}\eta}\cosh\{(2f/cH^2)^{\frac{1}{2}}(\eta - y)\}, \tag{22}$$

where A and A' are given respectively by

$$\begin{aligned} A\eta_1^{(\frac{1}{4}+2f/\kappa^2 c)^{\frac{1}{2}}-\frac{1}{2}}&[(2f/\kappa^2 c)^{\frac{1}{2}}\sinh\{(2f/cH^2)^{\frac{1}{2}}(\eta-\eta_1)\} \\ &\quad + \{(\tfrac{1}{4}+2f/\kappa^2 c)^{\frac{1}{2}} - \tfrac{1}{2}\}\cosh\{(2f/cH^2)^{\frac{1}{2}}(\eta-\eta_1)\}] \\ &= -gsH/(f-\kappa^2 c)\,[(2f/\kappa^2 c)^{\frac{1}{2}}\sinh\{(2f/cH^2)^{\frac{1}{2}}(\eta-\eta_1)\} + \cosh\{(2f/cH^2)^{\frac{1}{2}}(\eta-\eta_1)\}] \\ &\quad + gsH/f\,(2f/\kappa^2 c)^{\frac{1}{2}}\sinh\{(2f/cH^2)^{\frac{1}{2}}(\eta-\eta_1)\}, \end{aligned} \tag{23}$$

$$\begin{aligned} 2A'\, e^{(2f/cH^2)^{\frac{1}{2}}\eta}&\left[\cosh\{(2f/cH^2)^{\frac{1}{2}}(\eta-\eta_1)\} + \frac{(2f/\kappa^2 c)^{\frac{1}{2}}}{(\frac{1}{4}+2f/\kappa^2 c)^{\frac{1}{2}}-\frac{1}{2}}\sinh\{(2f/cH^2)^{\frac{1}{2}}(\eta-\eta_1)\}\right] \\ &= -gsH\frac{f - \{(\frac{1}{4}+2f/\kappa^2 c)^{\frac{1}{2}}-\frac{1}{2}\}\kappa^2 c}{f(f-\kappa^2 c)\{(\frac{1}{4}+2f/\kappa^2 c)^{\frac{1}{2}}-\frac{1}{2}\}}. \end{aligned} \tag{24}$$

8. Numerical estimation of the effect of eddy viscosity on different velocity profiles

Numerical estimation of the velocity profiles in particular cases is rendered difficult by the fact that reliable values of the coefficient of eddy viscosity are not available. However, we know (e.g. from Schlichting 1955, p. 406) that in a turbulent flow over a rough surface, the velocity profile is given by

$$\bar{u} = 2{\cdot}5\,\tau^{\frac{1}{2}} \log z + \text{const.}, \tag{25}$$

where τ is the drag per unit area of the surface and z is the height above the surface.

The value $\bar{\nu}$ of eddy viscosity in the neighbourhood of the surfaces is given by

$$\bar{\nu} = \frac{\tau}{d\bar{u}/dz} = \frac{fu^2z}{2{\cdot}5(fu^2)^{\frac{1}{2}}} = 0{\cdot}4f^{\frac{1}{2}}uz. \tag{26}$$

We assume that the expression (26) for $\bar{\nu}$ holds throughout the fluid, and hence the mean value ν of $\bar{\nu}$ is $0{\cdot}2f^{\frac{1}{2}}uh$. Hence, from the relation $\nu = chu$ used in § 6, we get that

$$c = 0{\cdot}2f^{\frac{1}{2}}. \tag{27}$$

While no strict accuracy is claimed for (27), especially as it assumes the eddy viscosities for vertical and horizontal momentum exchange to be equal, it is presumed that the value of c given by it is at least of the correct order of magnitude.

y/η ...	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
u/U	1·20	1·18	1·15	1·12	1·09	1·06	1·03	0·99	0·95	0·91	0·87
u/u_{max}	0	0·43	0·59	0·71	0·79	0·86	0·92	0·95	0·98	0·99	1·00

TABLE 2

y/η ...	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
u/U	1·04	1·04	1·04	1·04	1·04	1·03	1·02	1·01	0·99	0·96	0·92
u/u_{max}	0	0·36	0·51	0·62	0·71	0·79	0·86	0·92	0·96	0·99	1·00

TABLE 3

The value of f will depend on the nature of the bed and the hydraulic radius of the channel, and it is best determined in terms of Manning's coefficient n which is connected to f by the empirical relationships (see, for example, Rouse 1938, pp. 279 and 280)

$$f = \frac{g}{C^2}, \quad C = \frac{1{\cdot}486R^{\frac{1}{6}}}{n}, \tag{28}$$

where R is the hydraulic radius expressed in feet. For illustrative purposes we shall take $f = 0{\cdot}005$. For a channel of depth of about 20 ft. with a slope of one in a thousand, this coefficient corresponds to a bed of earth, gravel or rubble.

We consider a triangular channel with $f = 0{\cdot}005$ and $s = 0{\cdot}001$. Table 2 gives the results when $\kappa = \frac{1}{3}$, and table 3 gives the results when $\kappa = \frac{1}{6}$. In these tables, U denotes the mean velocity when eddy viscosity is neglected and u_{max} denotes the mid-stream velocity.

Similar calculations can be made for a trapezoidal channel. With $\kappa = \frac{1}{3}$, $f = 0{\cdot}005$, $s = 0{\cdot}001$, $H = 20$, $\eta = 2\eta_1 = 120$, the results given in table 4 are found.

The results show that the effects of eddy viscosity are important in regions where depth is changing, and in narrow channels. In wide rivers, its effects will be confined to near the walls.

y/η_1 ...	0	0·2	0·4	0·6	0·8	1·0
u/U	1·20	1·16	1·12	1·05	1·01	0·94
$\dfrac{y-\eta_1}{\eta-\eta_1}$	0	0·2	0·4	0·6	0·8	1·0
u/U	0·94	0·97	0·98	0·99	0·99	0·99

TABLE 4

The author wishes to acknowledge his gratitude to Prof. M. J. Lighthill for his helpful advice during the preparation of this paper, to the Nuffield Foundation for the grant of a Fellowship, and to the University of Dacca for leave of absence.

REFERENCES

MILNE-THOMSON, L. M. 1950 *Jacobian Elliptic Function Tables*. New York: Dover.
ROUSE, H. 1938 *Fluid Mechanics for Hydraulic Engineers*. New York: McGraw Hill.
SCHLICHTING, H. 1955 *Boundary Layer Theory*. London: Pergamon.
STOKER, J. J. 1957 *Water Waves*. New York: Interscience.